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STRESS DISTRIBUTION IN CONCENTRICALLY-HOLLOWED THICK-WALLED TUBES SUBJECTED TO UNIFORM RADIAL LOADING

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There are several industrial processes whereby thick-walled tubes are subjected to uniformly distributed radial stresses in which part of the tube's wall thickness undergoes plastic deformation, while the rest of it remains elastic. The most widely known and intensely studied of these processes is autofrettage. Autofrettage is a process whereby a thick-walled tube is subjected to internal pressurization until, hopefully, an inner sleeve undergoes plastic deformation, while an outer sleeve remains elastic. Although the investigators of this process are seeking to determine the stress distribution after depressurization, the stress distribution of the tubes under pressure and the corresponding pressure have to be determined in order to arrive at the retained stress distribution. In general, when uniformly distributed radial stresses are acting on the outer (diametrical) surface of a thick-walled tube, the tangential stress component (throughout the tube's wall thickness) has the same sign as the radial component. If, however, uniformly distributed radial stresses are acting on the inner surface of the same tube, the tangential and the radial stress components will be of the opposite sign to each other. In an elastically deformed tube, the magnitude of the tangential stress component increases towards the inner surface regardless of whether the imposed radial stress is at the outer or the inner surface, as well as with increasing magnitude of the imposed radial stress. However, if loaded at the tube's interior after plastic deformation commences, the magnitude of the tangential component decreases (from a maximum at the elastic-plastic interface) towards the tube's inner surface and in very large wall thickness tubes it might reverse its sign (at some intermediary radius between the elastic-plastic interface and the inner surface) assuming the same sign as that of the imposed radial component. In a more generalized form, the computational method used in autofrettage analysis can be utilized in the analysis of other related processes and/or products. A press-fitted concentric liner in a thick-walled tube is such an example. While the outer tube is subjected to uniformly distributed radial stresses at its bore, the liner is subjected to the same radial stresses at its outer diameter. During heat treatment of tubular components, the cooled annulus imposes a uniformly distributed radial stress on the uncooled sleeve or liner and vice versa. The equations for the calculation of the elastic-plastic interface diameter and the stresses on that surface and the equations for the stress distribution thereof in both the outer sleeve and in the plastically deformed inner sleeve of the tube are presented in this report. The calculations of the imposed radial stress(es) (internal and/or external) responsible for such a distribution are also presented here. These equations have been derived on the assumption that the material is nonstrain-hardening and isotropic and that Mises' yield criterion prevails throughout the plastic region.

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INTRODUCTION

Autofrettage is a manufacturing process whereby a thick-walled cylindrical pressure vessel is pressurized well beyond its elastic limit. This pressurization causes the tube's interior to undergo plastic deformation. Under such a loading, the radial component of the stress throughout the tube's wall thickness is compressive, while the tangential component is mainly tensile. In very thick-walled tubes with a significant plastically deformed inner sleeve, the tangential stress component near the bore may also become compressive. However, upon depressurization, both the radial and the tangential components of the stress near the tube's bore are compressive. Likewise, the stress distribution throughout the wall thickness of a press-fitted liner inside a thick-walled tube is compressive for both the radial and tangential components.

It has long been established that autofrettage of a pressure vessel, before putting it into service, increases its fatigue life. However, the optimal amount of autofrettaging is not clear. Furthermore, it is not clear to this investigator how much of the improvement in fatigue life is due to the post-autofrettage compressive tangential stresses at the bore, and how much is due to the small plastic deformation that takes place near the bore under compressive hydrostatic stresses (during pressurization and possibly upon depressurization). The ability to compute the stress distribution during autofrettage and after depressurization can be useful in determining autofrettage optimization.

Furthermore, the ability to correlate the elastic-plastic interface and the state of stress on that surface, due to known stresses at the bore and/or radial stresses at the tube's outer diameter (O.D.), facilitates the means for predicting the interfacial radial stress between two concentrically press-fitted tubes (a tube and a liner). The derivations of equations required for the determination of the elastic-plastic interface and the state of stress on such a surface are presented here. Furthermore, the derivations of equations required for the determination of the stress distribution throughout the tube's plastic region are also presented.

EQUILIBRIUM, ELASTIC STRESS, AND YIELDING

The following is based on the assumption that in analyzing the stress distribution in the plastic regions of elastic-plastic (partially plastic) bodies, the constraint imposed by the elastically deformed portion (on the plastically deformed portion) can facilitate a useful added constraint not available in the analyses of most "large plastic deformation" processes (ref 1).

As shown in Figure 1 (ref 2), equilibrium prevails when

$$(\sigma_{rr} + d\sigma_{rr}) \cdot (r + dr) \cdot d\theta = -\sigma_{rr} \cdot d\theta + 2\sigma_{\theta\theta} \cdot dr \cdot \frac{d\theta}{2}$$

or

$$\begin{aligned} -\sigma_{rr} \cdot r \cdot d\theta + \sigma_{rr} \cdot dr \cdot d\theta + r \cdot d\sigma_{rr} \cdot d\theta + d\sigma_{rr} \cdot dr \cdot d\theta \\ = -\sigma_{rr} \cdot r \cdot d\theta + \sigma_{\theta\theta} \cdot r \cdot d\theta + \sigma_{\theta\theta} \cdot dr \cdot d\theta \end{aligned}$$

or

$$(r+dr)d\sigma_{rr} = (\sigma_{\theta\theta}-\sigma_{rr})dr$$

which for dr → 0 becomes

$$\frac{dr}{r} = \frac{d\sigma_{rr}}{\sigma_{\theta\theta}-\sigma_{rr}} \tag{1}$$

Furthermore, it is assumed here that equilibrium in the axial plane, Eq. (1), prevails at all times throughout the tube's wall thickness.

When a thick-walled tube is subjected to internal pressure, $p_i = -\sigma_{rr}$ @ r = a, and/or to external pressure, $p_o = -\sigma_{rr}$ @ r = b, at a level which preserves its elasticity, it has been shown that its elastic stress distribution is (ref 3)

$$\sigma_{\theta\theta} = \frac{\left[\left(\frac{b}{a}\right)^2 + \left(\frac{b}{r}\right)^2\right]\sigma_{rr(b)} - \left[\left(\frac{b}{r}\right)^2 + 1\right]\sigma_{rr(a)}}{\left(\frac{b}{a}\right)^2 - 1} \tag{2a}$$

and

$$\sigma_{rr} = \frac{\left[\left(\frac{b}{a}\right)^2 - \left(\frac{b}{r}\right)^2\right]\sigma_{rr(b)} + \left[\left(\frac{b}{r}\right)^2 - 1\right]\sigma_{rr(a)}}{\left(\frac{b}{a}\right)^2 - 1} \tag{2b}$$

These equations, known as Lamé's equations, automatically satisfy the equation of equilibrium, Eq. (1), in the elastic region, a ≥ r ≥ b.

It can be shown (ref 4) that if the stress, $\sigma_{rr(a)}$ or $\sigma_{rr(b)}$, at either of the tube's physical boundaries (r = a or r = b, respectively), is replaced by a known internal radial stress component, $\sigma_{rr(d)}$, at a surface r = d within the tube's body, a ≤ d ≤ b, so that either the inner sleeve, a ≤ r ≤ d, and/or the outer sleeve, d ≤ r ≤ b, preserves its elasticity, then Eqs. (2a) and (2b) can be replaced by Eqs. (3a) and (3b) and/or Eqs. (3c) and (3d), respectively.

$$\sigma_{\theta\theta} = \frac{\left[\left(\frac{d}{a}\right)^2 + \left(\frac{d}{r}\right)^2\right]\sigma_{rr(d)} - \left[\left(\frac{d}{r}\right)^2 + 1\right]\sigma_{rr(a)}}{\left(\frac{d}{a}\right)^2 - 1} \tag{3a}$$

and

$$\sigma_{rr} = \frac{\left[\left(\frac{d}{a}\right)^2 - \left(\frac{d}{r}\right)^2\right]\sigma_{rr(d)} + \left[\left(\frac{d}{r}\right)^2 - 1\right]\sigma_{rr(a)}}{\left(\frac{d}{a}\right)^2 - 1} \tag{3b}$$

in the elastic range a ≤ r ≤ d, and/or

$$\sigma_{\theta\theta} = \frac{\left[\left(\frac{b}{d}\right)^2 + \left(\frac{b}{r}\right)^2\right]\sigma_{rr(b)} - \left[\left(\frac{b}{r}\right)^2 + 1\right]\sigma_{rr(d)}}{\left(\frac{b}{d}\right)^2 - 1} \tag{3c}$$

and

$$\sigma_{rr} = \frac{\left[\left(\frac{b}{d}\right)^2 - \left(\frac{b}{r}\right)^2\right]\sigma_{rr(b)} + \left[\left(\frac{b}{r}\right)^2 - 1\right]\sigma_{rr(d)}}{\left(\frac{b}{d}\right)^2 - 1} \tag{3d}$$

in the elastic range d ≤ r ≤ b.

Assuming that Mises' yield criterion prevails in the plastic region, and in the absence of shear due to symmetry, then (ref 1)

$$\sqrt{\tfrac{1}{2}[(\sigma_{\theta\theta}-\sigma_{rr})^2 + (\sigma_{rr}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{\theta\theta})^2]} = \sigma_o \tag{4}$$

where $\sigma_o \equiv$ the material's yield strength, under uniaxial stress, in such a region.

In plane-stress, $\sigma_{zz} = 0$, and according to Hooke's law, Lamé's equations yield a uniform axial strain, ϵ_{zz}, throughout the elastic region.

$$\epsilon_{zz(r)} = \frac{1}{E}[\sigma_{zz(r)} - \nu(\sigma_{rr(r)}+\sigma_{\theta\theta(r)})] = -2\nu\frac{\left(\frac{b}{a}\right)^2 \cdot \sigma_{rr(b)} - \sigma_{rr(a)}}{\left[\left(\frac{b}{a}\right)^2 - 1\right] \cdot E} \tag{5}$$

which is independent of the coordinate (of the point in question). Furthermore, Mises' yield criterion in plane-stress can be rewritten as

$$\sigma_{\theta\theta}^2 - \sigma_{rr} \cdot \sigma_{\theta\theta} + \sigma_{rr}^2 = \sigma_o^2 \tag{6a}$$

Applying Lamé's equations, Eqs. (2a) and (2b), at r = a, together with Mises' yield criterion in plane-stress, Eq. (6a), yielding commences at the tube's bore when

$$\sigma_{rr(a)} = \frac{\left[3\left(\frac{b}{a}\right)^2 + 1\right]^2 \left(\frac{b}{a}\right)^2 + \sigma_{rr(b)} \pm \left[\left(\frac{b}{a}\right)^2 - 1\right]\sqrt{\left[3\left(\frac{b}{a}\right)^4 + 1\right]\sigma_o^2 - 3\left(\frac{b}{a}\right)^4 + \sigma_{rr(b)}}}{3\left(\frac{b}{a}\right)^4 + 1} \tag{7a}$$

which, in the absence of radial stresses at the tube's O.D., $\sigma_{rr(b)} = 0$, becomes

$$\sigma_{rr(a)} = \pm \frac{\left(\frac{b}{a}\right)^2 - 1}{\sqrt{3\left(\frac{b}{a}\right)^4 + 1}} \sigma_o \tag{8a}$$

and in the absence of radial stresses at the tube's bore, $\sigma_{rr(a)} = 0$, yields

$$\sigma_{rr(b)} = \pm \frac{\left(\frac{b}{a}\right)^2 - 1}{2\left(\frac{b}{a}\right)^2} \sigma_o \tag{9a}$$

as the radial stress at the tube's respective boundaries, when plastic deformation commences at the tube's bore, r = a.

Applying either $\sigma_{rr(a)}$ from Eq. (8a) or $\sigma_{rr(b)}$ from Eq. (9a) to Lamé's equations, Eqs. (2a) and (2b) (with the other value being zero, $\sigma_{rr(b)} = 0$ or $\sigma_{rr(a)} = 0$, respectively), at any arbitrary radius, $a \leq r \leq b$, results in

$$\sigma_{\theta\theta(r)}^2 - \sigma_{rr(r)} \cdot \sigma_{\theta\theta(r)} + \sigma_{rr(r)}^2 \leq \sigma_o^2$$

which means that by applying radial stress, at either the tube's bore or O.D., to the point of commencement of plastic deformation (yielding) at the bore, r=a, the remaining tube wall, $a \leq r \leq b$, remains elastic.

As stated above, when the stress distribution derived from Lamé's equations, Eqs. (2a) and (2b), is applied to Hooke's law, the result is a uniform axial strain distribution. Conversely, if the tube is constrained from being deformed axially, $\epsilon_{zz} = 0$, then the resultant axial stress in

the elastic regions of the tube is uniform

$$\sigma_{zz} = \nu(\sigma_{\theta\theta} + \sigma_{rr}) = 2\nu \frac{\left(\frac{b}{a}\right)^2 \cdot \sigma_{rr(b)} - \sigma_{rr(a)}}{\left(\frac{b}{a}\right)^2 - 1} \tag{10}$$

Applying σ_{zz} from Eq. (10) to Mises' yield criterion, as expressed in Eq. (4), yields

$$(1-\nu+\nu^2)\sigma_{\theta\theta}^2 - (1+2\nu-2\nu^2)\sigma_{rr} \cdot \sigma_{\theta\theta} + (1-\nu+\nu^2)\sigma_{rr}^2 = \sigma_o^2 \tag{6b}$$

as the relation between the radial and the tangential stress components, σ_{rr} and $\sigma_{\theta\theta}$, and the material's yield strength (in uniaxial loading), σ_o, at the elastic-plastic interface. Furthermore, in the computations that follow, it is assumed that Eq. (6b) prevails throughout the plastic zone in plane-strain (axially constrained tube).

By applying Lamé's equations, Eqs. (2a) and (2b), at r=a together with Mises' yield criterion in plane-strain, Eq. (6b), yielding commences at the tube's bore when

$$\sigma_{rr(a)} = \frac{\left[3\left(\frac{b}{a}\right)^2 + (1-2\nu)^2\right]\left(\frac{b}{a}\right)^2 \sigma_{rr(b)} \pm \left[\left(\frac{b}{a}\right)^2 - 1\right]\sqrt{\left[3\left(\frac{b}{a}\right)^4 + (1-2\nu)^2\right]\sigma_o^2 - 3(1-2\nu)^2\left(\frac{b}{a}\right)^4 \sigma_{rr(b)}^2}}{3\left(\frac{b}{a}\right)^4 + (1-2\nu)^2} \tag{7b}$$

which, in the absence of radial stress at the tube's O.D., $\sigma_{rr(b)} = 0$, becomes

$$\sigma_{rr(a)} = \pm \frac{\left(\frac{b}{a}\right)^2 - 1}{\sqrt{3\left(\frac{b}{a}\right)^4 + (1-2\nu)^2}} \sigma_o \tag{8b}$$

and, in the absence of radial stress at the tube's bore, $\sigma_{rr(a)} = 0$, yields

$$\sigma_{rr(b)} = \pm \frac{\left(\frac{b}{a}\right)^2 - 1}{2 \cdot \sqrt{1-\nu+\nu^2} \cdot \left(\frac{b}{a}\right)^2} \sigma_o \tag{9b}$$

as the radial stress at the tube's respective radial boundaries when plastic deformation commences at the tube's bore, r=a.

As in the case of plane-stress, applying either $\sigma_{rr(a)}$ from Eq. (8b) or $\sigma_{rr(b)}$ from Eq. (9b) to Lamé's equations, Eqs. (2a) and (2b), (with the other value being zero, $\sigma_{rr(b)} = 0$ or $\sigma_{rr(a)} = 0$, respectively) at any arbitrary radius, $a \le r \le b$, results in

$$(1-\nu+\nu^2)\sigma_{\theta\theta(r)}^2 - (1+2\nu+2\nu^2)\sigma_{rr(r)} \cdot \sigma_{\theta\theta(r)} + (1-\nu+\nu^2)\sigma_{rr(r)}^2 \le \sigma_o^2$$

which means that by applying radial stresses at either the tube's bore or O.D. to the point of commencement of plastic deformation at the bore, r=a, the remaining wall, $a \le r \le b$, remains elastic.

ELASTIC-PLASTIC INTERFACE

If, however,

$$|\sigma_{rr(a)}| > \frac{\left|\left[3\left(\frac{b}{a}\right)^2+1\right]\left(\frac{b}{a}\right)^2+\left[\left(\frac{b}{a}\right)^2-1\right]\sqrt{\left[3\left(\frac{b}{a}\right)^4+1\right]\left(\frac{\sigma_o}{\sigma_{rr(b)}}\right)^2-3\left(\frac{b}{a}\right)^4}\right|}{3\left(\frac{b}{a}\right)^4+1} \cdot |\sigma_{rr(b)}| \tag{10a}$$

in plane-stress, or

$$|\sigma_{rr(a)}| > \frac{\left|\left[3\left(\frac{b}{a}\right)^2+(1-2\nu)^2\right]\left(\frac{b}{a}\right)^2+\left[\left(\frac{b}{a}\right)^2-1\right]\sqrt{\left[3\left(\frac{b}{a}\right)^4+(1-2\nu)^2\right]\left(\frac{\sigma_o}{\sigma_{rr(b)}}\right)^2-3(1-2\nu)\left(\frac{b}{a}\right)^4}\right|}{3\left(\frac{b}{a}\right)^4+(1-2\nu)^2} \cdot |\sigma_{rr(b)}| \tag{10b}$$

in plane-strain; or conversely, if

$$|\sigma_{rr(b)}| > \frac{\left|\left[3\left(\frac{b}{a}\right)^2+1\right] + \left[\left(\frac{b}{a}\right)^2-1\right]\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(a)}}\right)^2-1}\right|}{4\left(\frac{b}{a}\right)^2}\cdot|\sigma_{rr(a)}| \tag{11a}$$

in plane-stress, or

$$|\sigma_{rr(b)}| > \frac{\left|\left[3\left(\frac{b}{a}\right)^2+(1-2\nu)^2\right] + \left[\left(\frac{b}{a}\right)^2-1\right]\sqrt{\frac{4}{3}\cdot\frac{1-\nu+\nu^2}{(1-2\nu)^2}\left(\frac{\sigma_o}{\sigma_{rr(a)}}\right)^2-1}\right|}{4(1-\nu+\nu^2)\left(\frac{b}{a}\right)^2}\cdot|\sigma_{rr(a)}| \tag{11b}$$

in plane-strain, then there is a radius $r=\rho$, where $a < \rho \leq b$, at which Lamé's equations (Eqs. (2a) and (2b)) prevail, while the yield criterion (Mises' in this case) is also satisfied. Thus, the cylindrical surface, $r=\rho$, is an elastic-plastic interface; the material in the sleeve $\rho \leq r \leq b$ is elastic and the region $a \leq r \leq \rho$ is plastic and satisfies the yield criterion.

For the case where there is no radial stress at the tube's O.D., $\sigma_{rr(b)} = 0$, Eqs. (10a) and (10b) are reduced to

$$|\sigma_{rr(a)}| > \frac{\left(\frac{b}{a}\right)^2 - 1}{\sqrt{3\left(\frac{b}{a}\right)^4 + 1}}\,\sigma_o$$

in plane-stress and to

$$|\sigma_{rr(a)}| > \frac{\left(\frac{b}{a}\right)^2 - 1}{\sqrt{3\left(\frac{b}{a}\right)^4 + (1-2\nu)^2}}\,\sigma_o$$

in plane-strain, and conversely in the absence of radial stresses at the tube's bore, $r=a$, $\sigma_{rr(a)} = 0$. Eqs. (11a) and (11b) are reduced to

$$\left|\sigma_{rr(b)}\right| > \frac{\left(\frac{b}{a}\right)^2 - 1}{2\left(\frac{b}{a}\right)^2}\,\sigma_o$$

in plane-stress, and to

$$\left|\sigma_{rr(b)}\right| > \frac{\left(\frac{b}{a}\right)^2 - 1}{2\cdot\sqrt{1-\nu+\nu^2}\cdot\left(\frac{b}{a}\right)^2}$$

in plane-strain.

By applying the selected yield criterion (Mises in this case) to Lamé's solution at the elastic-plastic interface, $r=\rho$, the result is

$$\sigma_{rr(\rho)} = \frac{\left[3\left(\frac{b}{\rho}\right)^2+1\right]\left(\frac{b}{a}\right)^2+\left[\left(\frac{b}{\rho}\right)^2-1\right]\sqrt{\left[3\left(\frac{b}{\rho}\right)^4+1\right]\left(\frac{\sigma_o}{\sigma_{rr(b)}}\right)^2-3\left(\frac{b}{\rho}\right)^4}}{3\left(\frac{b}{\rho}\right)^4+1}\cdot\sigma_{rr(b)} \tag{12a}$$

in plane-stress, and

$$\sigma_{rr(\rho)} = \frac{\left[3\left(\frac{b}{\rho}\right)^2+(1-2\nu)^2\right]\left(\frac{b}{\rho}\right)^2+\left[\left(\frac{b}{\rho}\right)^2-1\right]\sqrt{\left[3\left(\frac{b}{\rho}\right)^4+(1-2\nu)^2\right]\left(\frac{\sigma_o}{\sigma_{rr(b)}}\right)^2-3(1-2\nu)^2\left(\frac{b}{\rho}\right)^4}}{3\left(\frac{b}{\rho}\right)^4+(1-2\nu)^2}\cdot\sigma_{rr(b)} \tag{12b}$$

in plane-strain, for the radial component of the stress at the elastic-plastic interface, $r=\rho$.

Applying the value of $\sigma_{rr(\rho)}$, obtained through Eq. (12a) for a plane-stress problem, or $\sigma_{rr(\rho)}$, obtained through Eq. (12b) for a plane-strain problem, into Eqs. (3c) and (3d) (where $r=d$ is replaced by $r=\rho$), yields the stress distribution in the tube's elastic region, $\rho \le r \le b$. As previously stated, Lamé's equations automatically satisfy the equation of equilibrium, Eq. (1).

THE PLASTIC REGION

In the plastic region, $a \le r \le \rho$, however, the equation of equilibrium is the basis for the calculation of the stress distribution (refs 5,6). In order to solve Eq. (1), the value of the stress difference, $\sigma_{\theta\theta(r)} - \sigma_{rr(r)}$, has to be expressed in terms of the radial stress, $\sigma_{rr(r)}$. Assuming that Tresca's yield criterion prevails (ref 7), then $\sigma_{\theta\theta} - \sigma_{rr}$ = constant and the solution is simply

$$\ln \frac{r}{\rho} = \frac{1}{\sigma_o}\left[\sigma_{rr(r)} - \sigma_{rr(\rho)}\right]$$

where the value of $\sigma_{rr(\rho)}$ is applied as the known boundary condition at $r=\rho$. Here, however, it is assumed that Mises' yield criterion prevails in the plastic region, $a \le r \le \rho$. Thus, according to Eqs. (6a) and (6b), respectively

$$\sigma_{\theta\theta(r)} = \frac{\sigma_{rr(r)} \pm \sqrt{4\sigma_o^2 - 3\sigma_{rr(r)}^2}}{2} \qquad \textbf{(13a)}$$

in plane-stress and

$$\sigma_{\theta\theta(r)} = \frac{(1+2\nu-2\nu^2)\sigma_{rr(r)} \pm \sqrt{4(1-\nu+\nu^2)\sigma_o^2 - 3(1-2\nu)^2\sigma_{rr(r)}^2}}{2(1-\nu+\nu^2)} \qquad \textbf{(13b)}$$

in plane-strain or

$$\sigma_{\theta\theta(r)} - \sigma_{rr(r)} = -\frac{\sigma_{rr(r)} \mp \sqrt{4\sigma_o^2 - 3\sigma_{rr(r)}^2}}{2} \qquad \textbf{(13c)}$$

in plane-stress and

$$\sigma_{\theta\theta(r)} - \sigma_{rr(r)} = -\frac{(1-2\nu)^2\sigma_{rr(r)} \mp \sqrt{4(1-\nu+\nu^2)\sigma_o^2 - 3(1-2\nu)^2\sigma_{rr(r)}^2}}{2(1-\nu+\nu^2)} \qquad \textbf{(13d)}$$

in plane-strain.

From Eq. (3c),

$$\left|\sigma_{rr(b)}\right| > \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} \cdot \left|\sigma_{rr(\rho)}\right|$$

then $\sigma_{\theta\theta(\rho)}$ and $\sigma_{rr(\rho)}$ have the same sign. While if

$$\left|\sigma_{rr(b)}\right| < \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} \left|\sigma_{rr(\rho)}\right|$$

then $\sigma_{\theta\theta(\rho)}$ has the opposite sign of $\sigma_{rr(\rho)}$. Furthermore, $\ln \frac{r}{\rho} \leq 0$ in the range $a \leq r \leq \rho$, while $|\sigma_{rr(r)}|$ decreases with decreasing r when

$$\left|\sigma_{rr(b)}\right| > \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} \left|\sigma_{rr(\rho)}\right|$$

and increases with decreasing r when

$$\left|\sigma_{rr(b)}\right| < \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} \left|\sigma_{rr(\rho)}\right|$$

Equations (13c) and (13d) can be rewritten as

$$\sigma_{\theta\theta(r)} - \sigma_{rr(r)} = -\frac{1-(-1)^n \cdot \sqrt{4\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 3}}{2} \cdot \sigma_{rr(r)} \qquad \textbf{(14a)}$$

in plane-stress, or

$$\sigma_{\theta\theta(r)} - \sigma_{rr(r)} = - \frac{(1-2\nu)^2 - (-1)^n \cdot \sqrt{4(1-\nu+\nu^2) \cdot \left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 3(1-2\nu)^2}}{2(1-\nu+\nu^2} \cdot \sigma_{rr(r)} \qquad \textbf{(14b)}$$

in plane-strain, where n = 1 when

$$|\sigma_{rr(b)}| < \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} \cdot |\sigma_{rr(\rho)}|$$

and n = 2 when

$$|\sigma_{rr(b)}| < \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} \cdot |\sigma_{rr(\rho)}|$$

Thus, for the case where the radial loading of the bore, $\sigma_{rr(c)}$, dominates or where

$$|\sigma_{rr(b)}| < \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} \cdot |\sigma_{rr(\rho)}|$$

Eq. (14a) is applied to Eq. (1) (the equation of equilibrium), the result is

$$\frac{dr}{r} = -2 \cdot \frac{d\sigma_{rr}}{\sigma_{rr} + \sqrt{4\sigma_o^2 - 3\sigma_{rr}^2}} \qquad \textbf{(1a)}$$

Mises' solution (refs 5,6), as presented in Appendix A of this report, is

$$\ln\left(\frac{r}{\rho}\right) = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 1} + 1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2} - \ln\frac{4\left(\frac{b}{\rho}\right)^4}{3\left(\frac{b}{\rho}\right)^4 + 1}\right.$$

$$\left. - 2\sqrt{3}\left[\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 1} - \tan^{-1}\frac{3\left(\frac{b}{\rho}\right)^2 + 1}{\sqrt{3}\left[\left(\frac{b}{\rho}\right)^2 - 1\right]}\right]\right\} \tag{15a}$$

in plane-stress.

In plane-strain, applying Eq. (14b) to Eq. (1) results in

$$\frac{dr}{r} = -2\frac{(1-\nu+\nu^2)d\sigma_{rr}}{\left[(1-2\nu)^2 + \sqrt{4(1-\nu+\nu^2)\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 3(1-2\nu)^2}\right]\sigma_{rr}}$$

Letting $\delta = 1\text{-}v+v^2$ and $\eta = (1\text{-}2v)^2 = 1\text{-}4v+4v^2$, the above can be written as

$$\frac{dr}{r} = -2\frac{\delta\cdot d\sigma_{rr}}{\left[\eta + \sqrt{4\delta\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 3\eta}\right]\sigma_{rr}} \tag{1b}$$

and the solution, as shown in Appendix B, is

$$\ln\left(\frac{r}{\rho}\right) = -\frac{1}{4}\left\{\ln\frac{\left[\frac{\sqrt{3}}{\eta}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 1} + 1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2} - \ln\frac{4\delta\left(\frac{b}{\rho}\right)^4}{3\left(\frac{b}{\rho}\right)^4 + \eta}\right.$$

$$\left. - 2\sqrt{\frac{3}{\eta}}\left[\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 1} - \tan^{-1}\frac{3\left(\frac{b}{\rho}\right)^2 + \eta}{\sqrt{3\eta}\left[\left(\frac{b}{\rho}\right)^2 - 1\right]}\right]\right\} \tag{15b}$$

If, however,

$$|\sigma_{rr(b)}| > \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2}|\sigma_{rr(\rho)}|$$

then

$$\sigma_{\theta\theta(\rho)} \cdot \sigma_{rr(\rho)} > 0$$

(or the tangential and the radial components of stress at the elastic-plastic interface have the same sign), for which the exponent n = 2 in Eqs. (14a) and (14b). Thus, applying Eq. (14a) to Eq. (1) yields

$$\frac{dr}{r} = -2\frac{d\sigma_{rr}}{\left[1 - \sqrt{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 3}\right]\sigma_{rr}} \tag{1c}$$

The solution, as shown in Appendix C, is

$$\ln\left(\frac{r}{\rho}\right) = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1}-1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2} - \ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}-1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2}\right.$$

$$\left. + 2\sqrt{3}\left[\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 1} - \tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2 - 1}\right]\right\} \tag{15c}$$

for the cases where

$$|\sigma_{rr(b)}| \geq \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2}\,|\sigma_{rr(\rho)}|$$

in plane-stress.

Similarly, in plane-strain under radial compressive stress, and when the tangential component, $\sigma_{\theta\theta(\rho)}$, at the elastic-plastic interface, has the same sign as the radial component of the stress, $\sigma_{rr(\rho)}$ then

$$\sigma_{\theta\theta} = \frac{(1+2\nu-2\nu^2)\sigma_{rr} - \sqrt{4(1-\nu+\nu^2)\sigma_o^2 - 3(1-2\nu)^2\sigma_{rr}^2}}{2(1-\nu+\nu^2)}\,\frac{\sigma_{rr}}{|\sigma_{rr}|}$$

only in the narrow range of

$$\frac{\sigma_o}{\sqrt{1-\nu+\nu^2}} \leq |\sigma_{rr}| \leq \frac{2\sqrt{1-\nu+\nu^2}}{\sqrt{3}(1-2\nu)}\cdot\sigma_o$$

However,

$$\sigma_{\theta\theta} = \frac{(1+2\nu-2\nu^2)\sigma_{rr} + \sqrt{4(1-\nu+\nu^2)\sigma_o^2 - 3(1-2\nu)^2\sigma_{rr}^2}}{2(1-\nu+\nu^2)}\,\frac{\sigma_{rr}}{|\sigma_{rr}|}$$

satisfies these conditions wherever

$$|\sigma_{rr}| \le 2\frac{\sqrt{1-\nu+\nu^2}}{\sqrt{3}(1-2\nu)} \cdot \sigma_o$$

Applying the above value of $\sigma_{\theta\theta}$ into Eq. (14b) with the exponent n = 2 and (with the substitution of $\delta = 1\text{-}v\text{+}v^2$ and $\eta = (1\text{-}2v)^2 = (1\text{-}4v\text{+}4v^2)$, into the equation of equilibrium, Eq. (1), yields

$$\frac{dr}{r} = -2\frac{\delta}{\eta}\frac{\delta d\sigma_{rr}}{\left[\eta - \sqrt{4\delta\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 3\eta}\right]\sigma_{rr}} \tag{1d}$$

The solution of Eq. (1d), as shown in Appendix D, is

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4}{3}\frac{\delta}{\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2} - \ln\frac{\left[\sqrt{\frac{3}{\eta}}\sqrt{\frac{4}{3}\frac{\delta}{\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2}\right.$$

$$\left. + 2\frac{\sqrt{3}}{\eta}\left[\tan^{-1}\sqrt{\frac{4}{3}\frac{\delta}{\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2 - 1} - \tan^{-1}\sqrt{\frac{4}{3}\frac{\delta}{\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2 - 1}\right]\right\} \tag{15d}$$

for the cases of

$$|\sigma_{rr(b)}| \ge \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2}|\sigma_{rr(\rho)}|$$

in plane-strain.

Depending on whether the stress at the bore, $\sigma_{rr(a)}$, or at the tube's O.D., $\sigma_{rr(b)}$, dominates the stress distribution (namely, whether $\sigma_{rr(\rho)} \bullet \sigma_{\theta\theta(\rho)} < 0$, or $\sigma_{rr(\rho)} \bullet \sigma_{\theta\theta(\rho)} > 0$, respectively) in plane-stress or in plane-strain, respectively, Eqs. (15a), (15b), (15c), or (15d) are used to determine the radial component of stress as a function of its radial location, r, within the plastic region $a \leq r \leq \rho$. The corresponding tangential component of stress is computed by using the corresponding Eqs. (13a), (13b), (13c), or (13d).

Equations (15a) and (15c) for plane-stress and Eqs. (15b) and (15d) for plane-strain can be combined respectively, as follows:

In plane-stress,

$$\ln\left(\frac{r}{\rho}\right) = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{4\sigma_o^2-3\sigma_{rr(r)}^2} - (-1)^n\cdot\sigma_{rr(r)}\right]^2}{4\sigma_o^2} - \ln\frac{\left[\sqrt{4\sigma_o^2-3\sigma_{rr(\rho)}^2} - (-1)^n\cdot\sigma_{rr(\rho)}\right]^2}{4\sigma_o^2} + 2\cdot\sqrt{3}\cdot(-1)^n\cdot\left[\tan^{-1}\frac{\sqrt{4_o^2-3\sigma_{rr(r)}^2}}{\sqrt{3}\,|\sigma_{rr(r)}|} - \tan^{-1}\frac{\sqrt{4\sigma_o^2-3\sigma_{rr(\rho)}^2}}{\sqrt{3}\cdot|\sigma_{rr(\rho)}|}\right]\right\} \qquad \textbf{(16a)}$$

where n = 1 for $\sigma_{rr} \bullet \sigma_{\theta\theta} < 0$ and n = 2 for $\sigma_{rr} \bullet \sigma_{\theta\theta} > 0$ and where according to Eq. (12a)

$$\sigma_{rr(\rho)} = \frac{\left[3\left(\frac{b}{\rho}\right)^2+1\right]\left(\frac{b}{\rho}\right)^2 + \left[\left(\frac{b}{\rho}\right)^2-1\right]\cdot\sqrt{\left[3\left(\frac{b}{\rho}\right)^4+1\right]\left(\frac{\sigma_o}{\sigma_{rr(b)}}\right)^2 - 3\left(\frac{b}{\rho}\right)^4}}{\sqrt{3}\left(\frac{b}{\rho}\right)^4+1} \cdot \sigma_{rr(b)}$$

from which, when loaded at the bore only, namely when $\sigma_{rr} \bullet \sigma_{\theta\theta} < 0$ and $\sigma_{rr(b)} = 0$ becomes

$$|\sigma_{rr(\rho)}| = \frac{\left(\frac{b}{\rho}\right)^2 - 1}{\sqrt{3\left(\frac{b}{\rho}\right)^4+1}}$$

and

$$\frac{\left[\sqrt{4\sigma_o^2 - 3\sigma_{rr(\rho)}} - (-1)^n \sigma_{rr(\rho)}\right]^2}{4\sigma_o^2} = \frac{4\left(\frac{b}{\rho}\right)^4}{3\left(\frac{b}{\rho}\right)^4 + 1}$$

and

$$\frac{\sqrt{4\sigma_o^2 - 3\sigma_{rr(\rho)}}}{\sqrt{3}\cdot|\sigma_{rr(\rho)}|} = \frac{3\left(\frac{b}{\rho}\right)^2 + 1}{\sqrt{3}\left[\left(\frac{b}{\rho}\right)^2 - 1\right]}$$

CONCLUSIONS

The levels of uniform radial stresses, either at the tube's bore or O.D., when plastic deformation commences under plane-stress or plane-strain conditions have been established. Similarly, with a known uniform radial stress at the tube's O.D., the radial stress at any given radial surface, $r=\rho$ (for any arbitrary radius, $a \le \rho < b$) corresponding to plastic yielding at that surface has been established in plane-stress as well as in plane-strain.

Equations that correlate the radial stress component with its radial location, r, have been discussed. The actual derivations of these equations have been demonstrated for the cases where

$$|\sigma_{rr(b)}| < \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} |\sigma_{rr(\rho)}|$$

$$|\sigma_{rr(b)}| > \frac{\left(\frac{b}{\rho}\right)^2 + 1}{2\left(\frac{b}{\rho}\right)^2} |\sigma_{rr(\rho)}|$$

in plane-stress as well as in plane-strain. The details of these derivations, as they apply to the plastic region, $a \le r \le \rho$, are presented in the Appendices of this report. The corresponding tangential component is computed from the radial component and when plane-strain is considered, the corresponding axial component is computed from its two other orthogonal components.

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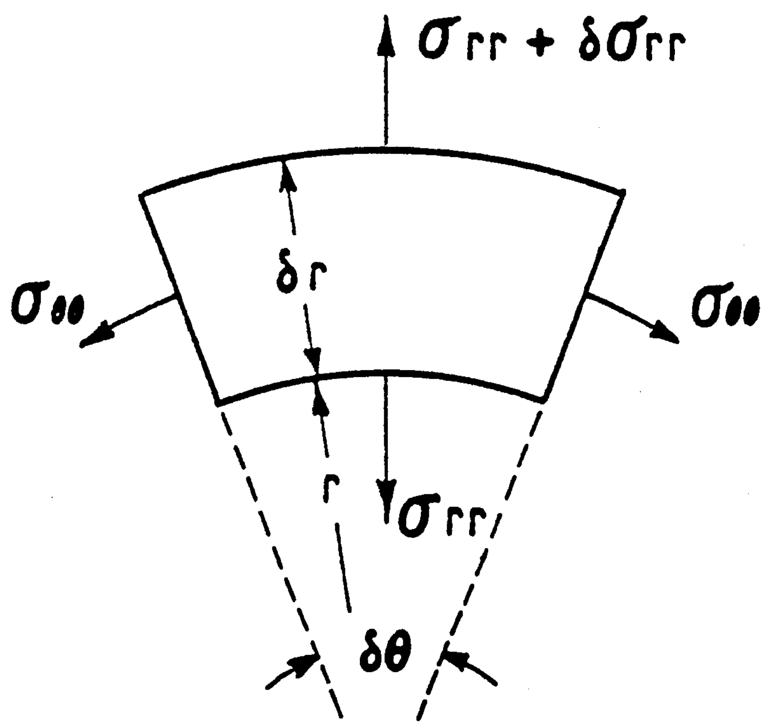

Figure 1. Stress equilibrium in a cylindrical shell.

APPENDIX A

In order to solve the integral

$$-2\int_{\rho}^{r} \frac{d\sigma_{rr}}{\left[1+\sqrt{3}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right]}\,\sigma_{rr}$$

as in Appendix C, let

$$\sigma_{rr}^2 = \frac{\frac{4}{3}\cdot\sigma_o^2}{t^2+1}$$

where

$$t = \sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

Therefore,

$$\sigma_{rr} = \frac{\frac{2}{\sqrt{3}}\sigma_o}{\sqrt{t^2+1}}$$

and

$$d\sigma_{rr} = \frac{2}{\sqrt{3}}\sigma_o \cdot d\frac{1}{\sqrt{t^2+1}}$$

Let

$$t^2 + 1 = s^2$$

then

$$2t \cdot dt = 2s \cdot ds$$

or

$$\frac{ds}{dt} = \frac{t}{s} = \frac{t}{\sqrt{t^2+1}}$$

and

$$\frac{d}{dt}\left(\frac{1}{\sqrt{t^2+1}}\right) = \frac{d}{ds}\left(\frac{1}{s}\right) \cdot \frac{ds}{dt} = -\frac{1}{s^2} \cdot \frac{t}{\sqrt{t^2+1}} = -\frac{t}{\sqrt{(t^2+1)^3}}$$

Hence,

$$d\sigma_{rr} = -\frac{2 \cdot \sigma_o \cdot t}{\sqrt{3} \cdot \sqrt{(t^2+1)^3}} \cdot dt$$

and

$$\begin{aligned}
\sigma_{rr} + \sqrt{4\sigma_o^2 - 3\sigma_{rr}^2} &= \sigma_{rr} - \sqrt{3} \cdot \sigma_{rr} \cdot \sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} \\
&= \sigma_{rr} \cdot \left[1 + \sqrt{3} \cdot \sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1}\right] \\
&= \frac{\frac{2}{\sqrt{3}} \cdot \sigma_o}{\sqrt{t^2+1}} [1 + \sqrt{3}t]
\end{aligned}$$

Thus,

$$-2\frac{d\sigma_{rr}}{\sigma_{rr} + \sqrt{4\sigma_o^2 - 3\sigma_{rr}^2}} = +\frac{4 \cdot \sigma_o \cdot t}{\sqrt{3} \cdot \sqrt{(t^2+1)^3} \cdot \frac{2}{\sqrt{3}} \cdot \frac{\sigma_o}{\sqrt{t^2+1}} \cdot [1 + \sqrt{3} \cdot t]} \cdot dt$$

$$= +2 \frac{t}{(\sqrt{3}t + 1 \cdot (t^2+1)} \cdot dt$$

and

$$-2\int \frac{d\sigma_{rr}}{\sigma_{rr}+\sqrt{4\sigma_o^2-3\sigma_{rr}^2}} = 2\int \frac{t}{(\sqrt{3}t+1)\ \cdot\ (t^2+1)}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\left\{\int\frac{\sqrt{3}t+1}{(\sqrt{3}t+1)\ \cdot\ (t^2+1)}\,dt - \int \frac{dt}{\sqrt{3}t+1)\ \cdot\ (t^2+1)}\right\}$$

$$= \frac{2}{\sqrt{3}}\left\{\int\frac{dt}{t^2+1} - \int\frac{dt}{(\sqrt{3}t+1)\ \cdot\ (t^2+1)}\right\}$$

where

$$\int\frac{dt}{t^2+1} = \tan^{-1}t = \tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1}$$

where

$$z^2 = \frac{3}{\eta}t^2 + 2\sqrt{\frac{3}{\eta}}t+1 = \left[\sqrt{\frac{3}{\eta}}t+1\right]^2 = \left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2$$

and

$$z^2-2z+4\frac{\delta}{\eta} = \left[\frac{3}{\eta}t+2\sqrt{\frac{3}{\eta}}t+1\right]-2\left[\sqrt{\frac{3}{\eta}}t+1\right]+4\frac{\delta}{\eta} = \frac{3}{\eta}t^2+\frac{4\delta-\eta}{\eta} = \frac{3}{\eta}(t^2+1)$$

$$= \frac{3}{\eta}\cdot\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 = 4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2$$

and

$$z-1 = \sqrt{\frac{3}{\eta}}t = \sqrt{\frac{3}{\eta}}\ \cdot\ \sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

Thus,

$$\int \frac{dt}{\left(\sqrt{\frac{3}{\eta}}t+1\right)\cdot(t^2+1)} = \frac{1}{8\frac{\delta}{\eta}}\sqrt{\frac{3}{\eta}}\cdot\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} + \frac{1}{4\frac{\delta}{\eta}}\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

$$= \frac{\sqrt{3}\eta}{8\delta}\cdot\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} + \frac{2}{\sqrt{\frac{3}{\eta}}}\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right\}$$

from which

$$-2\frac{\delta}{\eta}\int\frac{d\sigma_{rr}}{\sigma_{rr}+\sqrt{\frac{1}{\eta}}\cdot\sqrt{4\frac{\delta}{\eta}\sigma_o^2-3\sigma_{rr}^2}} = \frac{2\delta}{3\eta}\left\{\int\frac{dt}{t^2+1} - \int\frac{dt}{\left(\sqrt{\frac{3}{\eta}}t+1\right)(t^2+1)}\right\}$$

$$= \frac{2\delta}{3\eta}\cdot\left\{\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1} - \frac{1}{8\frac{\delta}{\eta}}\cdot\sqrt{\frac{3}{\eta}}\cdot\ln\frac{\left[\sqrt{\frac{3}{\eta}}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2}\right.$$

Whereas for

$$-\int\frac{dt}{(\sqrt{3}t+1)\cdot(t^2+1)}$$

let

$$\sqrt{3}t+1 = z$$

Then

$$\sqrt{3}\,dt = dz \quad , \quad t = \frac{z-1}{\sqrt{3}} \quad , \quad and \quad t^2+1 = \frac{z^2-2z+4}{3}$$

Thus,

$$\begin{aligned}
\int \frac{dt}{(\sqrt{3}t+1)\cdot(t^2+1)} &= \int \frac{3\cdot dz}{\sqrt{3}\cdot z\cdot(z^2-2z+4)} \\
&= \sqrt{3}\int \frac{dz}{z\cdot(z^2-2z+4} \\
&= \sqrt{3}\cdot\left\{\frac{1}{8}\ln\frac{z^2}{z^2-2z+4} + \frac{1}{4}\int\frac{dz}{z^2-2z+4}\right\} \\
&= \sqrt{3}\cdot\left\{\frac{1}{8}\ln\frac{z^2}{z^2-2z+4} + \frac{1}{4}\frac{2}{\sqrt{16-4}}\tan^{-1}\frac{2z-2}{\sqrt{16-4}}\right\}
\end{aligned}$$

where

$$\begin{aligned}
z^2 &= 3t^2 + 2\sqrt{3t}+1 \\
&= 3\left[\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1\right] + 2\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} + 1 \\
&= \left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} + 1\right]^2
\end{aligned}$$

and

$$\begin{aligned}
z^2-2z+4 &= (3t^2+2\sqrt{3}t+1) - (2\sqrt{3}t+2) + 4 \\
&= 3(t^2+1) = 4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2
\end{aligned}$$

Thus

$$\int \frac{dt}{(\sqrt{3}t+1)\cdot(t^2+1)} = \left\{\frac{\sqrt{3}}{8}\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} + 1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} + \frac{1}{4}\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1}\right\}$$

and

$$-2\int \frac{d\sigma_{rr}}{\sigma_{rr}+\sqrt{4\sigma_o^2-3\sigma_{rr}^2}} = \frac{2}{\sqrt{3}}\left\{\int \frac{dt}{t^2+1} - \int \frac{dt}{(\sqrt{3}t+1)\cdot(t^2+1)}\right\}$$

$$= \frac{2}{\sqrt{3}}\left\{\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1} - \frac{\sqrt{3}}{8}\cdot\ln\frac{\left[\sqrt{3}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - \frac{1}{4}\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right\}$$

$$= -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]}{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - 2\sqrt{3}\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right\}$$

or

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1}+1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2} - \ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}+1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2}\right.$$

$$\left. - 2\sqrt{3}\left[\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1} - \tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}\right]\right\}$$

Whereas, in Appendix C,

$$\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2 = \frac{3\left(\frac{b}{\rho}\right)^4+1}{\left[\left(\frac{b}{\rho}\right)^2-1\right]^2} \quad , \quad \frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1 = \frac{\left[3\left(\frac{b}{\rho}\right)^2+1\right]}{3\left[\left(\frac{b}{\rho}\right)-1\right]}$$

and where

$$\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}+1\right] = \frac{\left[3\left(\frac{b}{\rho}\right)^2+1\right]+\left[\left(\frac{b}{\rho}\right)^2-1\right]}{\left(\frac{b}{\rho}\right)^2-1} = \frac{4\left(\frac{b}{\rho}\right)^2}{\left(\frac{b}{\rho}\right)^2-1}$$

and

$$\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}+1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)} = \frac{\dfrac{16\left(\frac{b}{\rho}\right)^4}{\left[\left(\frac{b}{\rho}\right)^2-1\right]^2}}{4\cdot\dfrac{3\left(\frac{b}{\rho}\right)^4+1}{\left[\left(\frac{b}{\rho}\right)^2-1\right]^2}} = \frac{4\left(\frac{b}{\rho}\right)^4}{3\left(\frac{b}{\rho}\right)^4+1}$$

Hence, in plane-stress

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - \ln\frac{4\left(\frac{b}{\rho}\right)^4}{3\left(\frac{b}{\rho}\right)^4+1}\right.$$

$$\left.-2\sqrt{3}\left[\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1} - \tan^{-1}\frac{3\left(\frac{b}{\rho}\right)^2+1}{\sqrt{3}\left[\left(\frac{b}{\rho}\right)^2-1\right]}\right]\right\}$$

APPENDIX B

In order to solve the integral

$$-2\frac{\delta}{\eta}\int_{\rho}^{r}\frac{d\sigma_{rr}}{\left[1+\sqrt{\frac{3}{\eta}}\sqrt{\frac{4}{3}\frac{\delta}{\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right]\sigma_{rr}}$$

as in Appendix D, let

$$\sigma_{rr}^2 = \frac{\frac{4\delta}{3\eta}\cdot\sigma_o^2}{t^2+1}$$

where

$$t = \sqrt{\frac{4\delta}{3\eta}\cdot\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1}$$

then

$$\sigma_{rr} = \frac{\frac{4\delta}{3\eta}\cdot\sigma_o}{t^2+1}$$

and

$$d\sigma_{rr} = \sqrt{\frac{4\delta}{3\eta}}\cdot\sigma_o\cdot d\frac{1}{\sqrt{t^2+1}}$$

Let

$$t^2+1 = s^2,\ then\ 2t\cdot dt = 2s\cdot ds\ or\ \frac{ds}{dt} = \frac{t}{s} = \frac{t}{\sqrt{t^2+1}}$$

and

$$\frac{d}{dt}\left(\frac{1}{\sqrt{t^2+1}}\right) = \frac{d}{ds}\left(\frac{1}{s}\right)\cdot\frac{ds}{dt} = -\frac{1}{s^2}\cdot\frac{t}{\sqrt{t^2+1}} = -\frac{t}{\sqrt{(t^2+1)^3}}$$

Hence,

$$d\sigma_{rr} = \frac{\sqrt{\frac{4\delta}{3\eta}}\cdot\sigma_o\cdot t}{\sqrt{(t^2+1)^3}}\cdot dt$$

whereas

$$\sigma_{rr} - \sqrt{\frac{1}{\eta}}\cdot\sqrt{4\frac{\delta}{\eta}\sigma_o^2 - 3\sigma_{rr}^2} = \sigma_{rr} + \sqrt{\frac{3}{\eta}}\cdot\sigma_{rr}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1}$$

$$= \sigma_{rr}\left[1 + \sqrt{\frac{3}{\eta}}\cdot\frac{\sqrt{4\delta}}{3\eta}\cdot\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1\right]$$

$$= \frac{\sqrt{\frac{4\delta}{3\eta}}}{\sqrt{t^2+1}}\sigma_o\cdot\left[1 + \sqrt{\frac{3}{\eta}}\cdot t\right]$$

Thus,

$$-2\frac{\delta}{\eta}\cdot\frac{d\sigma_{rr}}{\sigma_{rr}+\sqrt{\frac{1}{\eta}}\cdot\sqrt{4\frac{\delta}{\eta}\sigma_o^2-3\sigma_{rr}^2}} = -2\frac{\delta}{\eta}\cdot\frac{\sqrt{t^2+1}\cdot\frac{\frac{\sqrt{4\delta}}{3\eta}}{\sqrt{(t^2+1)^3}}\cdot\sigma_o t}{\frac{\sqrt{4\delta}}{3\eta}\cdot\sigma_o\left[\sqrt{\frac{3}{\eta}}t+1\right]}\cdot dt$$

$$= -2\frac{\delta}{\eta}\frac{t}{(t^2+1)\cdot\left(\sqrt{\frac{3}{\eta}}t+1\right)}\cdot dt$$

or

$$-2\frac{\delta}{\eta}\int\frac{d\sigma_{rr}}{\sigma_{rr}+\sqrt{\frac{1}{\eta}}\cdot\sqrt{4\frac{\delta}{\eta}\sigma_o^2-3\sigma_{rr}^2}} = -2\frac{\delta}{\eta}\int\frac{t}{\left(\sqrt{\frac{3}{\eta}}t+1\right)\cdot(t^2+1)}\cdot dt$$

$$= -2\frac{\delta}{\sqrt{3\eta}}\left\{\int\frac{\sqrt{\frac{3}{\eta}}t+1}{\left(\sqrt{\frac{3}{\eta}}t+1\right)\cdot(t^2+1)}dt-\int\frac{dt}{\left(\sqrt{\frac{3}{\eta}}t+1\right)\cdot(t^2+1)}\right\}$$

$$= -2\frac{\delta}{\sqrt{3\eta}}\left\{\int\frac{dt}{t^2+1} + \int\frac{dt}{\left(\sqrt{\frac{3}{\eta}}t+1\right)\cdot(t^2+1)}\right\}$$

where

$$\int\frac{dt}{t^2+1} = \tan^{-1}t = \tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

Whereas for

$$\int \frac{dt}{\left(\sqrt{\frac{3}{\eta}}t+1\right)\cdot(t^2+1)}$$

let

$$\sqrt{\frac{3}{\eta}}t+1 = z$$

then

$$\sqrt{\frac{3}{\eta}}\cdot dt = dz$$

and

$$t = \frac{z-1}{\sqrt{\frac{3}{\eta}}}$$

Thus,

$$t^2+1 = \frac{(z-1)^2}{\left(\frac{3}{\eta}\right)}+1 = \frac{z^2-2z+\left(1+\frac{3}{\eta}\right)}{\left(\frac{3}{\eta}\right)} = \frac{z^2-2z+4\frac{\delta}{\eta}}{\left(\frac{3}{\eta}\right)}$$

since

$$3+\eta = 3+(1-4\nu+4\nu^2 = 4(1-\nu+\nu^2) = 4\delta$$

Thus,

$$\int \frac{dt}{\left(\sqrt{\frac{3}{\eta}}t+1\right)\cdot(t^2+1)} = \int \frac{\frac{3}{\eta}\cdot dz}{\sqrt{\frac{3}{\eta}}z\cdot\left(z^2-2z+4\frac{\delta}{\eta}\right)}$$

$$= \sqrt{\frac{3}{\eta}}\left\{\frac{1}{8\frac{\delta}{\eta}}\ln\frac{z^2}{z^2-2z+4\frac{\delta}{\eta}} + \frac{2}{8\frac{\delta}{\eta}}\int\frac{dz}{z^2-2z+4\frac{\delta}{\eta}}\right\}$$

$$= \sqrt{\frac{3}{\eta}}\left\{\frac{1}{8\frac{\delta}{\eta}}\ln\frac{z^2}{z^2-2z+4\frac{\delta}{\eta}} + \frac{1}{4\frac{\delta}{\eta}}\cdot\frac{2}{\sqrt{16\frac{\delta}{\eta}-4}}\tan^{-1}\frac{2z-2}{\sqrt{16\frac{\delta}{\eta}-4}}\right\}$$

$$= \frac{1}{8\frac{\delta}{\eta}}\cdot\sqrt{\frac{3}{\eta}}\ln\frac{z^2}{z^2-2z+4\frac{\delta}{\eta}} + \frac{1}{4\frac{\delta}{\eta}}\tan^{-1}\frac{z-1}{\sqrt{\frac{3}{\eta}}}$$

$$- \frac{1}{4\frac{\delta}{\eta}}\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

$$= -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - 2\sqrt{\frac{3}{\eta}}\cdot\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right\}$$

or

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1}+1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2} - \ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}+1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2}\right.$$

$$\left.-2\sqrt{\frac{3}{\eta}}\left[\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1}-\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}\right]\right\}$$

which for

$$\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2 = \frac{3\left(\frac{b}{\rho}\right)^4+(1-2\nu)^2}{\left[\left(\frac{b}{\rho}\right)-1\right]^2} \quad , \quad \frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1 = \frac{\left[3\left(\frac{b}{\rho}\right)^2+\eta\right]^2}{3\eta\left[\left(\frac{b}{\rho}\right)^2-1\right]^2}$$

and

$$\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1 = \frac{4\delta\left(\frac{b}{\rho}\right)^2}{\eta\left[\left(\frac{b}{\rho}\right)^2-1\right]}$$

becomes

$$\ln\frac{r}{\rho} = \frac{1}{4}\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}+1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - \ln\frac{4\delta\left(\frac{b}{\rho}\right)^4}{3\left(\frac{b}{\rho}\right)^4+\eta}\right.$$

$$\left. -2\sqrt{\frac{3}{\eta}}\left[\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1} - \tan^{-1}\frac{3\left(\frac{b}{\rho}\right)^2+\eta}{\sqrt{3\eta}\left[\left(\frac{b}{\rho}\right)^2-1\right]}\right]\right\}$$

APPENDIX C

In order to solve the integral

$$-2\int_{\rho}^{r} \frac{d\sigma_{rr}}{\left[1-\sqrt{3}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right]\sigma_{rr}}$$

let

$$\sigma_{rr}^2 = \frac{\frac{4}{3}\cdot\sigma_o^2}{t^2+1}$$

where

$$t = \sqrt{\frac{4}{3}\cdot\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

Therefore,

$$\sigma_{rr} = \frac{\frac{2}{\sqrt{3}}}{\sqrt{t^2+1}}\sigma_o$$

and

$$d\sigma_{rr} = \frac{2}{\sqrt{3}}\sigma_o \cdot d\frac{1}{\sqrt{t^2+1}}$$

Let

$$t^2+1 = s^2$$

then

$$2t\cdot dt = 2s\cdot ds$$

or

$$\frac{ds}{dt} = \frac{t}{s} = \frac{t}{\sqrt{t^2+1}}$$

and

$$\frac{d}{dt}\left(\frac{1}{\sqrt{t^2+1}}\right) = \frac{d}{ds}\left(\frac{1}{s}\right)\cdot\frac{ds}{dt} = \frac{1}{s^2}\cdot\frac{t}{\sqrt{t^2+1}} = -\frac{t}{\sqrt{(t^2+1)^3}}$$

Hence,

$$d\sigma_{rr} = -\frac{2\cdot\sigma_o\cdot t}{\sqrt{3}\cdot\sqrt{(t^2+1)^3}}\cdot dt$$

and

$$\begin{aligned}
\sigma_{rr} - \sqrt{4\sigma_o^2-3\sigma_{rr}^2} &= \sigma_{rr} - \sqrt{3}\cdot\sigma_{rr}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1} \\
&= \sigma_{rr}\cdot\left[1-\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right] \\
&= \frac{\frac{2}{\sqrt{3}}\cdot\sigma_o}{\sqrt{t^2+1}}\left[1-\sqrt{3}\cdot t\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
-2\frac{d\sigma_{rr}}{\sigma_{rr}-\sqrt{4\sigma_o^2-3\sigma_{rr}^2}} &= +\frac{4\cdot\sigma_o\cdot t}{\sqrt{3}\cdot\sqrt{(t^2+1)^3}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\sigma_o}{\sqrt{t^2+1}}\cdot[1-\sqrt{3}\cdot t]}\cdot dt \\
&= -2\frac{t}{(\sqrt{3}t-1)\cdot(t^2+1)}\cdot dt
\end{aligned}$$

and

$$-2\int\frac{d\sigma_{rr}}{\sigma_{rr}-\sqrt{4\sigma_o^2-3\sigma_{rr}^2}} = -2\int\frac{t}{(\sqrt{3}t-1)\cdot(t^2+1)}dt$$

$$= \frac{-2}{\sqrt{3}}\left\{\frac{\sqrt{3}t-1}{(\sqrt{3}t-1)\cdot(t^2+1)}dt+\int\frac{dt}{(\sqrt{3}t-1)\cdot(t^2+1)}\right\}$$

$$= \frac{-2}{\sqrt{3}}\left\{\int\frac{dt}{t^2+1}+\int\frac{dt}{(\sqrt{3}t-1)\cdot(t^2+1)}\right\}$$

where

$$\int\frac{dt}{t^2+1} = \tan^{-1}t = \tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

Whereas for

$$\int\frac{dt}{(\sqrt{3}t-1)\cdot(t^2+1)}$$

Let

$$\sqrt{3}t-1 = z$$

Then

$$\sqrt{3}dt = dz \ , \ t = \frac{z+1}{\sqrt{3}}, and\ t^2+1 = \frac{z^2+2z+4}{3}$$

Thus,

$$\int\frac{dt}{(\sqrt{3}t-1)\cdot(t^2+1)} = \int\frac{3\cdot dz}{\sqrt{3}\cdot z\cdot(z^2+2z+4)}$$

$$= \sqrt{3}\int\frac{dz}{z\cdot(z^2+2z+4)}$$

$$= \sqrt{3}\cdot\left\{\frac{1}{8}\ln\frac{z^2}{z^2+2z+4}-\frac{1}{4}\int\frac{dz}{z^2+2z+4}\right\}$$

$$= \sqrt{3}\cdot\left\{\frac{1}{8}\ln\frac{z^2}{z^2+2z+4}-\frac{1}{4}\frac{2}{\sqrt{16-4}}\tan^{-1}\frac{2z+2}{\sqrt{16-4}}\right\}$$

where

$$
\begin{aligned}
z^2 &= 3t^2 - 2\sqrt{3}t + 1 \\
&= 3\left[\frac{4}{3}\cdot\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1\right] - 2\sqrt{3}\cdot\sqrt{\frac{4}{3}\cdot\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} + 1 \\
&= \left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} - 1\right]^2
\end{aligned}
$$

and

$$
\begin{aligned}
z^2\, 2z + 4 &= (3t^2 - 2\sqrt{3}t + 1) + (2\sqrt{3}t - 2) + 4 \\
&= 3t^2 + 3 \\
&= 3(t^2+1) = 4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2
\end{aligned}
$$

Therefore,

$$
\int \frac{dt}{(\sqrt{3}t-1)\cdot(t^2+1)} = \left\{\frac{\sqrt{3}}{8}\ln\frac{[\sqrt{3}t-1]^2}{3(t^2+1)} - \frac{1}{4}\tan^{-1}t\right\}
$$

and

$$
-2\int \frac{t}{(\sqrt{3}t-1)\cdot(t^2+1)}\,dt = -\frac{2}{\sqrt{3}}\left\{\tan^{-1}t + \sqrt{\frac{3}{8}}\cdot\ln\frac{[\sqrt{3}t-1]^2}{3(t^2+1)} - \frac{1}{4}\tan^{-1}t\right\}
$$

$$
= -\frac{1}{4}\left\{\ln\frac{[\sqrt{3}t-1]^2}{3(t^2+1)} + 2\sqrt{3}\cdot\tan^{-1}t\right\}
$$

$$
= -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} - 1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} + 2\sqrt{3}\cdot\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1}\right\}
$$

or

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(rr)}}\right)^2-1}-1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - \ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}-1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2}\right.$$

$$\left. + \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1} - \tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}\right]\right\}$$

However, for the cases where

$$\sigma_{rr}\cdot\sigma_{\theta\theta} > 0 \quad , \quad \sigma_{rr(b)} \neq 0$$

and hence one has to calculate

$$\sigma_{rr(\rho)} = f(\sigma_{rr(b)})$$

from the pertaining Eq. (12a) or (12b) before using the above equation (presented also as Eq. (15c) in the text).

$$\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2 = \frac{3\left(\frac{b}{\rho}\right)^4+1}{\left[\left(\frac{b}{\rho}\right)^2-1\right]^2}$$

and

$$\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1} = \frac{3\left(\frac{b}{\rho}\right)^2+1}{\sqrt{3}\left[\left(\frac{b}{\rho}\right)^2-1\right]}$$

Thus,

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{3}\cdot\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} - 1\right]^2}{4\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - \ln\frac{\left[\left(\frac{b}{\rho}\right)^2 + \right]^2}{3\left(\frac{b}{\rho}\right)^4 + 1}\right.$$

$$\left. + 2\sqrt{3}\left[\tan^{-1}\sqrt{\frac{4}{3}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1} - \tan^{-1}\frac{3\left(\frac{b}{\rho}\right)^2 + 1}{\sqrt{3}\left[\left(\frac{b}{\rho}\right)^2 - 1\right]}\right]\right\}$$

APPENDIX D

In order to solve the integral

$$-2\frac{\delta}{\eta}\int_{\rho}^{r}\frac{d\sigma_{rr}}{\left[1-\sqrt{\frac{3}{\eta}}\sqrt{\frac{4}{3}\frac{\delta}{\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right]\sigma_{rr}}$$

let

$$\sigma_{rr}^2 = \frac{\frac{4\delta}{3\eta}\cdot\sigma_o^2}{t^2+1}$$

where

$$t = \sqrt{\frac{4\delta}{3\eta}\cdot\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

then

$$\sigma_{rr} = \frac{\sqrt{\frac{4\delta}{3\eta}}}{\sqrt{t^2+1}}\sigma_o$$

and

$$d\sigma_{rr} = \sqrt{\frac{4\delta}{3\eta}}\cdot\sigma_o\cdot d\frac{1}{\sqrt{t^2+1}}$$

Let

$$t^2+1 = s^2 \quad , \quad then\ 2t\cdot dt = 2s\cdot ds\ or\ \frac{ds}{dt} = \frac{t}{s} = \frac{t}{\sqrt{t^2+1}}$$

and

$$\frac{d}{dt}\left(\frac{1}{\sqrt{t^2+1}}\right) = \frac{d}{ds}\left(\frac{1}{s}\right)\cdot\frac{ds}{dt} = -\frac{1}{s^2}\cdot\frac{t}{\sqrt{t^2+1}} = -\frac{t}{\sqrt{(t^2+1)^3}}$$

Hence,

$$d\sigma_{rr} = \frac{\sqrt{\frac{4\delta}{3\eta}}\cdot\sigma_o\cdot t}{\sqrt{(t^2+1)^3}}\cdot dt$$

whereas

$$\sigma_{rr}-\sqrt{\frac{1}{\eta}}\cdot\sqrt{4\frac{\delta}{\eta}\sigma_o^2-3\sigma_{rr}^2} = \sigma_{rr}-\sqrt{\frac{3}{\eta}}\cdot\sigma_{rr}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

$$= \sigma_{rr}\left[1-\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right]$$

$$= \frac{\sqrt{\frac{4\delta}{3\eta}}}{\sqrt{t^2+1}}\cdot\sigma_o\cdot\left[1-\sqrt{\frac{3}{\eta}}\cdot t\right]$$

Thus,

$$-2\frac{\delta}{\eta}\frac{d\sigma_{rr}}{\sigma_{rr}-\sqrt{\frac{1}{\eta}}\cdot\sqrt{4\frac{\delta}{\eta}\sigma_o^2-3\sigma_{rr}^2}} = \frac{2\cdot\frac{\delta}{\eta}\cdot\sqrt{t^2+1}\cdot\frac{\sqrt{\frac{4\delta}{3\eta}}\cdot\sigma_o t}{\sqrt{(t^2+1)^2}}}{\sqrt{\frac{4\delta}{3\eta}}\cdot\sigma_o\cdot\left[\sqrt{\frac{3}{\eta}}t-1\right]}$$

$$= \frac{2\frac{\delta}{\eta}\cdot t}{(t^2+1)\cdot\left(\sqrt{\frac{3}{\eta}}t-1\right)}$$

or

$$-2\frac{\delta}{\eta}\int\frac{d\sigma_{rr}}{\sigma_{rr}-\sqrt{\frac{1}{\eta}}\cdot\sqrt{4\frac{\delta}{\eta}\sigma_o^2-3\sigma_{rr}^2}} = -2\frac{\delta}{\eta}\int\frac{t}{\left(\sqrt{\frac{3}{\eta}}t-1\right)\cdot(t^2+1)}\cdot dt$$

$$= -2\frac{\delta}{\sqrt{3n}}\left\{\int\frac{\sqrt{\frac{3}{\eta}}t-1}{\left(\sqrt{\frac{3}{\eta}}t-1\right)\cdot(t^2+1)}dt + \int\frac{dt}{\left(\sqrt{\frac{3}{\eta}}t-1\right)\cdot(t^2+1)}\right\}$$

$$= -2\frac{\delta}{\sqrt{3\eta}}\left\{\int\frac{dt}{t^2+1} + \int\frac{dt}{\left(\sqrt{\frac{3}{\eta}}t-1\right)\cdot(t^2+1)}\right\}$$

where

$$\int\frac{dt}{t^2+1} = \tan^{-1}t = \tan^{-1}\sqrt{\frac{4\delta}{3\eta}\cdot\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

Whereas for

$$\int\frac{dt}{\left(\sqrt{\frac{3}{\eta}}t-1\right)\cdot(t^2+1)}$$

Let

$$\sqrt{\frac{3}{\eta}}t-1 = z$$

then

$$\sqrt{\frac{3}{\eta}}\cdot dt = dz$$

and

$$t = \frac{z+1}{\sqrt{\frac{3}{\eta}}}$$

Therefore,

$$t^2+1 = \frac{(z+1)^2}{\left(\frac{3}{\eta}\right)}+1 = \frac{z^2+2z+\left(1+\frac{3}{\eta}\right)}{\left(\frac{3}{\eta}\right)} = \frac{z^2+2z+\frac{3+\eta}{\eta}}{\left(\frac{3}{\eta}\right)} = \frac{z^2+2z+4\frac{\delta}{\eta}}{\left(\frac{3}{\eta}\right)}$$

since

$$3+\eta = 3+(1-4\nu+4\nu^2) = 4(1-\nu+\nu^2) = 4\delta$$

Thus,

$$\int \frac{dt}{\left(\sqrt{\frac{3}{\eta}}t-1\right)\cdot(t^2+1)} = \int \frac{dz}{\sqrt{\frac{3}{\eta}}\cdot z\cdot\frac{z^2+2z+4\frac{\delta}{\eta}}{\left(\frac{3}{\eta}\right)}}$$

$$= \sqrt{\frac{3}{\eta}}\cdot\int\frac{dz}{z[z^2+2z+4\frac{\delta}{\eta}]}$$

$$= \sqrt{\frac{3}{\eta}}\left\{\frac{1}{8\frac{\delta}{\eta}}\cdot\ln\frac{z^2}{z^2+2z+4\frac{\delta}{\eta}} - \frac{2}{8\frac{\delta}{\eta}}\cdot\int\frac{dz}{z^2+2z+4\frac{\delta}{\eta}}\right\}$$

$$= \frac{\sqrt{3\eta}}{8\delta}\left\{\ln\frac{z^2}{z^2+2z+4\frac{\delta}{\eta}} - \frac{4}{2\sqrt{\frac{3}{\eta}}}\tan^{-1}\frac{2z+2}{2\sqrt{\frac{3}{\eta}}}\right\}$$

where

$$z+1 = \sqrt{\frac{3}{\eta}}\,t = \sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2 - 1}$$

and

$$z^2+2z+4\frac{\delta}{\eta} = \left(\frac{3}{\eta}t^2 - 2\sqrt{\frac{3}{\eta}}\,t+1\right)+\left(2\sqrt{\frac{3}{\eta}}\,t-2\right)+4\frac{\delta}{\eta}$$

$$= \frac{3}{\eta}t^2 + \frac{4\delta-\eta}{\eta} = \frac{3}{\eta}(t^2+1) = 4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2$$

Thus,

$$\int \frac{dt}{\left(\sqrt{\frac{3}{\eta}}\,t-1\right)\cdot(t^2+1)} = \frac{\sqrt{3\eta}}{8\delta}\cdot\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - \frac{\eta}{4\delta}\cdot\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}$$

and

$$-2\frac{\delta}{\eta}\int\frac{t}{\left(\sqrt{\frac{3}{\eta}}\,t-1\right)\cdot(t^2+1)}\,dt = -\left\{\frac{2\delta}{\sqrt{3\eta}}\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1} + \frac{1}{4}\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2}\right.$$

$$\left. - \frac{1}{2\sqrt{\frac{3}{\eta}}}\cdot\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right\}$$

$$= -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} + 2\sqrt{\frac{3}{\eta}}\cdot\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}\right\}$$

since

$$\frac{2\delta}{\sqrt{3\eta}} - \frac{1}{2\sqrt{\frac{3}{\eta}}} = \frac{2\delta}{\sqrt{3\eta}}\left(1-\frac{\eta}{4\delta}\right) = \frac{2\delta}{\sqrt{3\eta}}\left(\frac{4\delta-\eta}{4\delta}\right) = \frac{1}{2}\cdot\sqrt{\frac{3}{\eta}}$$

or

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2} - \ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2}\right.$$

$$\left. + 2\sqrt{\frac{3}{\eta}}\left[\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(r)}}\right)^2-1} - \tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2-1}\right]\right\}$$

As the case of $\sigma_{rr} \bullet \sigma_{\theta\theta} > 0$, where $\sigma_{rr(b)} \neq 0$, in plane stress, here too one has to calculate $\sigma_{rr(\rho)} = f(\sigma_{rr(b)})$ from the pertaining Eq. (12b) before using the above equation (presented also as Eq. (15d) in the text).

$$\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2 = \frac{3\left(\frac{b}{\rho}\right)^4 + \eta}{\left[\left(\frac{b}{\rho}\right)^2 - 1\right]^2}$$

and for

$$\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr(\rho)}}\right)^2 - 1} = \frac{4\delta\left[3\left(\frac{b}{\rho}\right)^4 + \eta\right] - 3\eta\left[\left(\frac{b}{\rho}\right)^4 - 2\left(\frac{b}{\rho}\right)^2 + 1\right]}{3\eta\left[\left(\frac{b}{\rho}\right)^2 - 1\right]^2}$$

$$= \frac{\sqrt{3[4\delta - \eta]\left(\frac{b}{\rho}\right)^4 + 6\eta\left(\frac{b}{\rho}\right)^2 + \eta[4\delta - 3]}}{3\eta\left[\left(\frac{b}{\rho}\right)^2 - 1\right]}$$

$$= \frac{\sqrt{9\left(\frac{b}{\rho}\right)^4 + 6\eta\left(\frac{b}{\rho}\right)^2 + \eta^2}}{\sqrt{3\eta\left[\left(\frac{b}{\rho}\right)^2 - 1\right]}}$$

$$= \frac{3\left(\frac{b}{\rho}\right)^2 + \eta}{\sqrt{3\eta\left[\left(\frac{b}{\rho}\right)^2 - 1\right]}}$$

The relation between the radial stress, σ_{rr}, and its radial distance, r, complies with the following relation:

$$\ln\frac{r}{\rho} = -\frac{1}{4}\left\{\ln\frac{\left[\sqrt{\frac{3}{\eta}}\cdot\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1}-1\right]^2}{4\frac{\delta}{\eta^2}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2} - \ln\frac{\left[\frac{3-\eta}{2}\cdot\left(\frac{b}{\rho}\right)^2+\eta\right]}{\delta\left[3\left(\frac{b}{\rho}\right)^4+\eta\right]}\right.$$

$$\left. + 2\sqrt{\frac{3}{\eta}}\left[\tan^{-1}\sqrt{\frac{4\delta}{3\eta}\left(\frac{\sigma_o}{\sigma_{rr}}\right)^2-1} - \tan^{-1}\frac{3\left(\frac{b}{\rho}\right)2+\eta}{\sqrt{3\eta}\left[\left(\frac{b}{\rho}\right)^2-1\right]}\right]\right\}$$

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